Additive decompositions of Mueller matrices

Polarimetric subtraction

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Introduction. Main aim of the talk

On providing mathematical tools for exploiting polarimetric measurements

- Physical parameters inside a Mueller matrix
  - → *Mean transmittance*
  - → *Polarizance*
  - → *Diattenuation*
  - → *Depolarization index*
  - → *Indices of polarimetric purity*
  - → *Components of purity*
- Serial decompositions
- Parallel decompositions (Mueller – Stokes)
- Subtraction (Mueller – Stokes)
- …

Additive decompositions of Mueller matrices. Polarimetric subtraction

1. Concept of Mueller matrix
2. Parallel decompostions of a Mueller matrix
   - *Spectral*
   - *Trivial*
   - *Arbitrary*
3. Physical quantities in a Mueller matrix
4. Polarimetric subtraction

1

The concept of Mueller matrix

Characterization of Jones matrices

Linear passive system

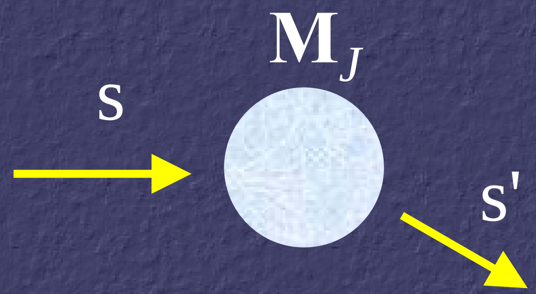
- Jones matrix $\mathbf{T}$ is a 2x2 complex matrix (7 physical parameters)

$$\mathbf{T} \equiv \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

- $\mathbf{T}$ satisfies the transmittance condition (maximum gain $\leq$ 1)

$$\frac{1}{2}\left\{ \mathrm{tr}\left(\mathbf{T}^{\dagger}\mathbf{T}\right) + \left[\left(\mathrm{tr}\left(\mathbf{T}^{\dagger}\mathbf{T}\right)\right)^{2} - 4\det\left(\mathbf{T}^{\dagger}\mathbf{T}\right) \right]^{1/2} \right\} \leq 1$$

Basic interaction: Stokes-Mueller description



Homogeneous deterministic sample

Stokes vector

$$\mathrm{s} \equiv \begin{bmatrix} I \\ IP\cos 2\varphi \cos 2\chi \\ IP\cos 2\varphi \sin 2\chi \\ IP\sin 2\varphi \end{bmatrix}$$

- I *intensity*
- P *degree of polarization*
- χ *azimuth*
- φ *ellipticity*

$$\mathbf{s}' = \mathbf{M}_J\,\mathbf{s}$$

Mueller-Jones matrix

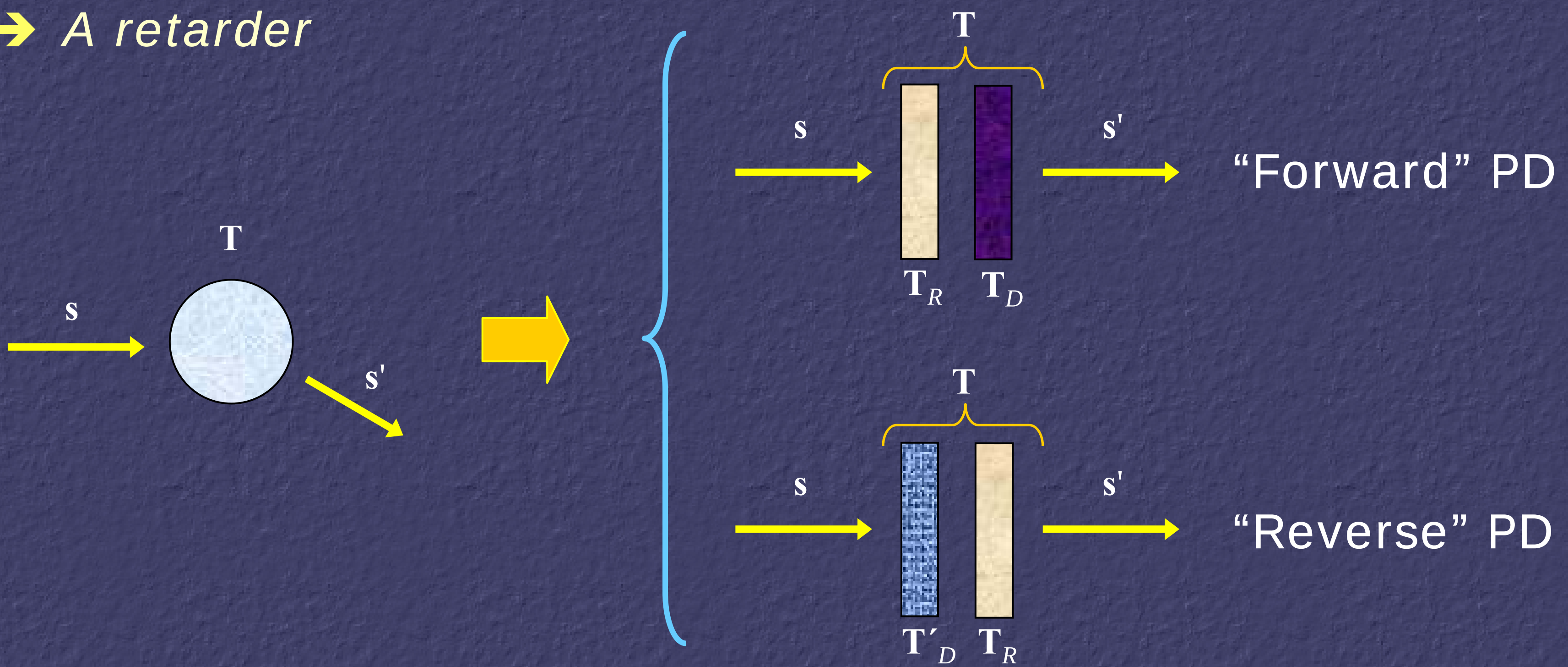
Incident beam: $P = 1$

Single interaction → $P' = 1$

The "pure case"

- Non-depolarizing (or pure) system: for incident light whit $P = 1$, emerging light has $P' = 1$

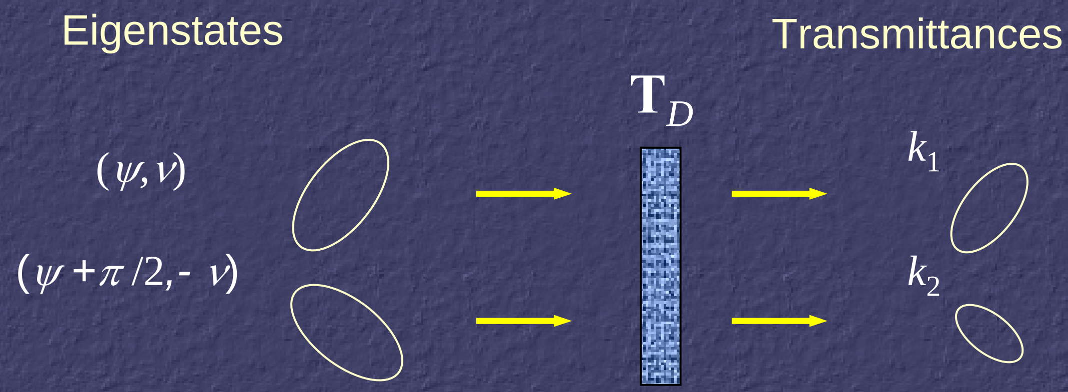
- The system is equivalent to a serial combination of two components (**polar decomposition**):
  - ➔ *A diattenuator (partial or total polarizer)*
  - ➔ *A retarder*



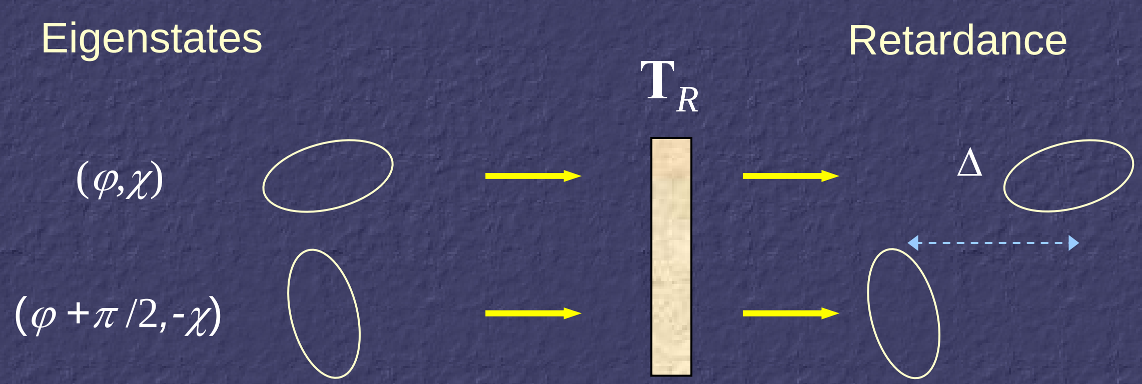
The "pure case"

7 independent physical quantities:

- **4** diattenuator



- **3** retarder



Mueller-Jones matrix

(or “pure Mueller matrix”)

$$\mathbf{M}_J(\mathbf{T}) = \mathbf{L}\left(\mathbf{T} \otimes \mathbf{T}^*\right)\mathbf{L}^{-1}$$

$$\mathbf{L} \equiv \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix}$$

Characterization of Mueller-Jones matrices

- *7 free parameters in* $\mathbf{T}$ *(Jones matrix)* $\Rightarrow$ *7 free parameters in*

$$\mathbf{M}_J\left(\mathbf{T}\right)=\mathbf{L}\left(\mathbf{T}\otimes\mathbf{T}^*\right)\mathbf{L}^{-1}$$

- *1 Transmittance condition*

$$m_{00}\left(1+D\right)\leq 1 \qquad \left(P=D\right)$$
$$m_{00}\left(1+P\right)\leq 1$$

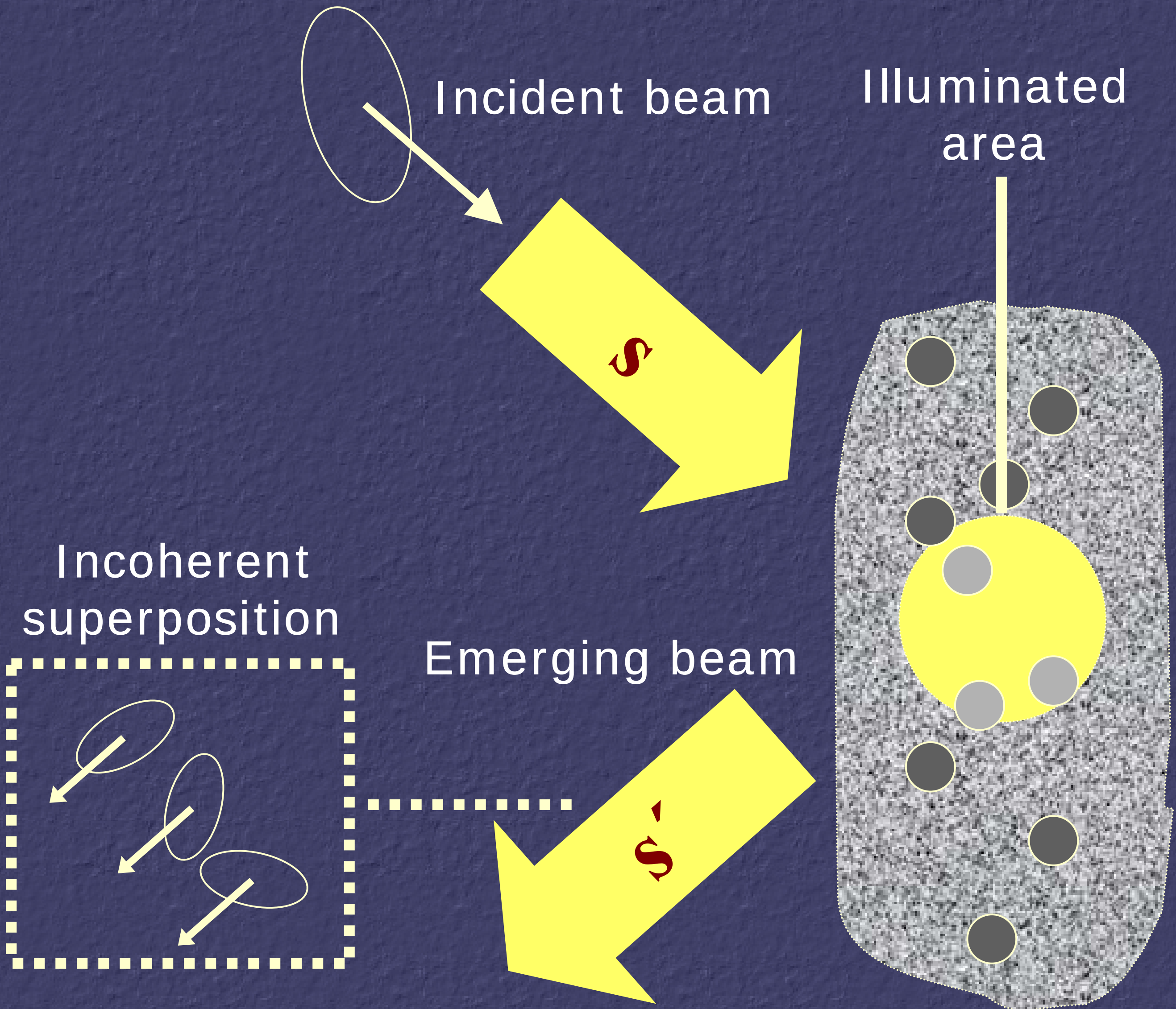
Note that

- The only kind of nondepolarizing system for arbitrary partially polarized input states of polarized light is a retarder (4 free parameters for a non-transparent retarder)

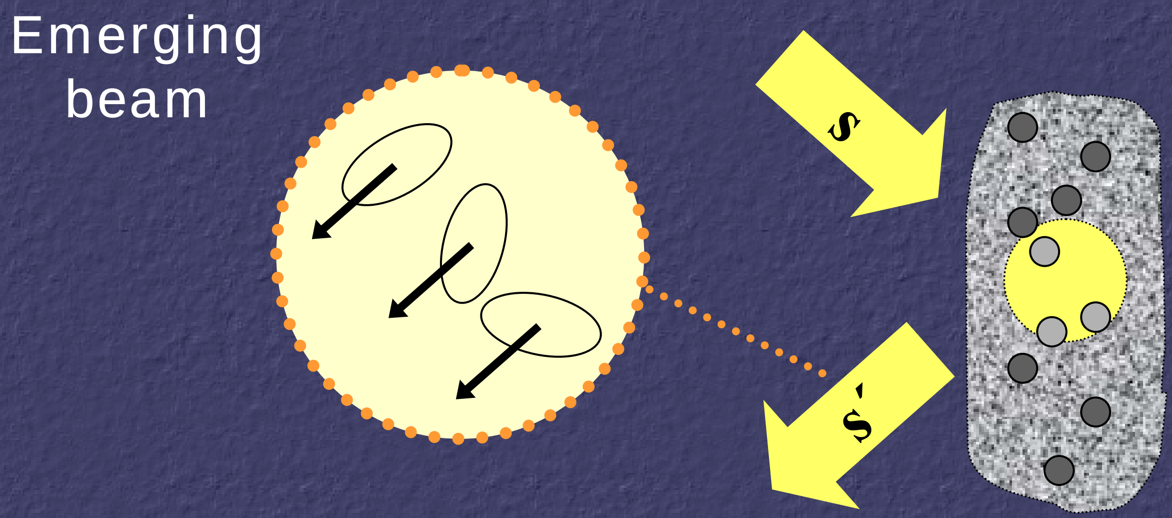
- Pure diattenuators (partial polarizers) depolarize some input partially polarized states $\Rightarrow$ polarizing-diattenuating properties are sources of certain depolarizing properties

General macroscopic interaction:

Synthesis of a Mueller matrix



Composed Mueller matrix



$$\mathbf{s}' = \sum_i \mathbf{s}'_i = \sum_i \mathbf{M}_{Ji} p_i \mathbf{s} = \underbrace{\sum_i p_i \mathbf{M}_{Ji}}_{\mathbf{M}} \mathbf{s}$$

$$\mathbf{M}_{Ji} \equiv \mathbf{L}(\mathbf{T}_i \otimes \mathbf{T}_i^*)\mathbf{L}^{-1}, \quad p_i \geq 0, \quad \sum_i p_i = 1$$

Partitioned form of a Mueller matrix

(for general Mueller matrices)

$$\mathbf{M} = m_{00}\begin{pmatrix} 1 & \mathbf{D}^T \\ \mathbf{P} & \mathbf{m} \end{pmatrix}$$

m_{00} *mean transmittance*

$$\mathbf{D} \equiv \frac{1}{m_{00}}\begin{pmatrix} m_{01} \\ m_{02} \\ m_{03} \end{pmatrix}$$

diattenuation vector

$D \equiv |\mathbf{D}|$ *diattenuation*

$$\mathbf{P} \equiv \frac{1}{m_{00}}\begin{pmatrix} m_{10} \\ m_{20} \\ m_{30} \end{pmatrix}$$

polarizance vector

$P \equiv |\mathbf{P}|$ *polarizance*

$$\mathbf{m} \equiv \frac{1}{m_{00}}\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

Depolarization index

J. J. Gil, E. Bernabéu. Opt. Acta **33**(2), 185-189 (1986)

$$P_\Delta = \sqrt{\frac{D^2 + P^2 + 3P_S^2}{3}}$$

$$P_S^2 \equiv \frac{1}{3}\|\mathbf{m}\|_2^2$$

$$P_\Delta = 1 \text{ "}\textit{nondepolarizing system}\text{"}$$

$$P_\Delta = 0 \text{ "}\textit{ideal depolarizer}\text{"}: P = D = P_S = 0$$

$$P_S : \text{"}\textit{Degree of spherical purity}\text{"}$$

$$P_S = 1 \ \textit{pure retarder}$$

$$P_S = 0 \ \textit{zero retardance...}$$

Covariance matrix associated with a Mueller matrix

$$\mathbf{M} \Longleftrightarrow \mathbf{H}$$

Covariance matrix $\mathbf{H}$

$\mathbf{H}$ *represents univocally the Mueller matrix and vice-versa*

$$\mathbf{H} = \frac{1}{4}\sum_{k,l=0}^{3} m_{kl}\mathbf{E}_{kl}$$

$$\mathbf{E}_{kl} = \boldsymbol{\sigma}_k \otimes \boldsymbol{\sigma}_l^*$$

$\boldsymbol{\sigma}_{kl}$ set of 4 "Pauli matrices"

$\mathbf{E}_{kl}$ set of 16 "Dirac matrices"

Coefficients m_{kl} are 16 measurable quantities: the 16 elements of the Mueller matrix $\mathbf{M}$ associated with $\mathbf{H}$

Covariance matrix H(M)

$$\mathbf{H}=\frac{1}{4}\begin{pmatrix}
\begin{matrix} m_{00}+m_{01} \\ +m_{10}+m_{11} \end{matrix} & \begin{matrix} m_{02}+m_{12} \\ +i\left(m_{03}+m_{13}\right) \end{matrix} & \begin{matrix} m_{20}+m_{21} \\ -i\left(m_{30}+m_{31}\right) \end{matrix} & \begin{matrix} m_{22}+m_{33} \\ +i\left(m_{23}-m_{32}\right) \end{matrix} \\
\begin{matrix} m_{02}+m_{12} \\ -i\left(m_{03}+m_{13}\right) \end{matrix} & \begin{matrix} m_{00}-m_{01} \\ +m_{10}-m_{11} \end{matrix} & \begin{matrix} m_{22}-m_{33} \\ -i\left(m_{23}+m_{32}\right) \end{matrix} & \begin{matrix} m_{20}-m_{21} \\ -i\left(m_{30}-m_{31}\right) \end{matrix} \\
\begin{matrix} m_{20}+m_{21} \\ +i\left(m_{30}+m_{31}\right) \end{matrix} & \begin{matrix} m_{22}-m_{33} \\ +i\left(m_{23}+m_{32}\right) \end{matrix} & \begin{matrix} m_{00}+m_{01} \\ -m_{10}-m_{11} \end{matrix} & \begin{matrix} m_{02}-m_{12} \\ +i\left(m_{03}-m_{13}\right) \end{matrix} \\
\begin{matrix} m_{22}+m_{33} \\ -i\left(m_{23}-m_{32}\right) \end{matrix} & \begin{matrix} m_{20}-m_{21} \\ +i\left(m_{30}-m_{31}\right) \end{matrix} & \begin{matrix} m_{02}-m_{12} \\ -i\left(m_{03}-m_{13}\right) \end{matrix} & \begin{matrix} m_{00}-m_{01} \\ -m_{10}+m_{11} \end{matrix}
\end{pmatrix}$$

Characterization of Mueller matrices

J .J. Gil, J. Opt. Soc. Am. **17**, 328—334 (2000)

- 4 Eigenvalue Conditions

$$0 \le \lambda_i , \; i = 0, 1, 2, 3$$

- 2 Transmittance Conditions

$$m_{00}\left(1 + D\right) \le 1$$

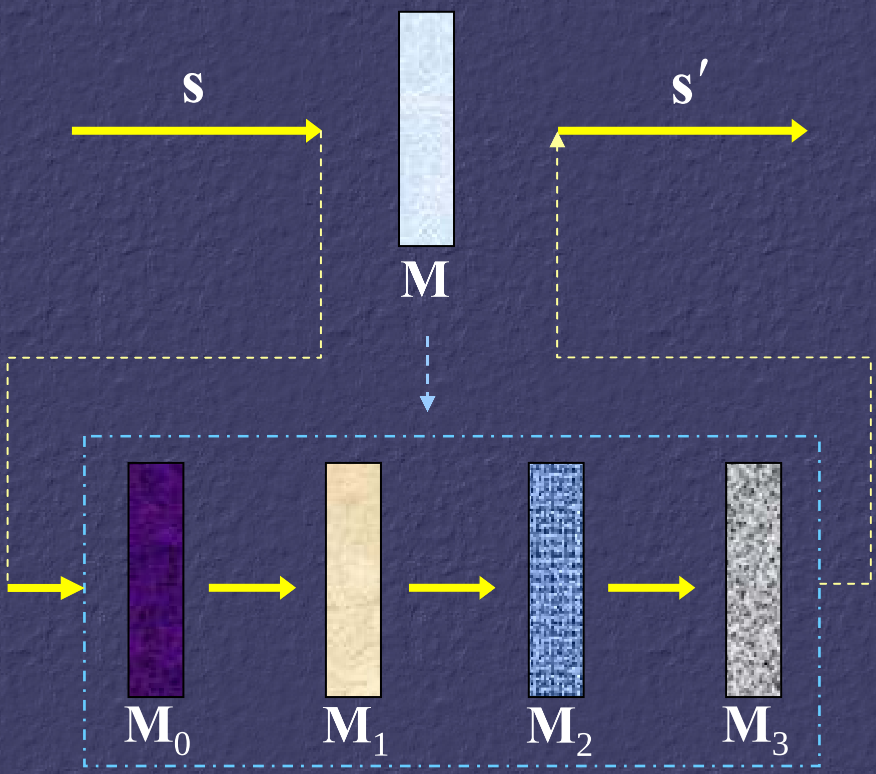
$$m_{00}\left(1 + P\right) \le 1$$

M (H)

$$\mathbf{M} = \begin{pmatrix} h_{00}+h_{11} & h_{00}-h_{11} & h_{01}+h_{10} & -i\left(h_{01}-h_{10}\right) \\ +h_{22}+h_{33} & +h_{22}-h_{33} & +h_{23}+h_{32} & -i\left(h_{23}-h_{32}\right) \\ & & & \\ h_{00}+h_{11} & h_{00}-h_{11} & h_{01}+h_{10} & -i\left(h_{01}-h_{10}\right) \\ -h_{22}-h_{33} & -h_{22}+h_{33} & -h_{23}-h_{32} & +i\left(h_{23}-h_{32}\right) \\ & & & \\ h_{02}+h_{20} & h_{02}+h_{20} & h_{03}+h_{30} & -i\left(h_{03}-h_{30}\right) \\ +h_{13}+h_{31} & -h_{13}-h_{31} & +h_{12}+h_{21} & +i\left(h_{12}-h_{21}\right) \\ & & & \\ i\left(h_{02}-h_{20}\right) & i\left(h_{02}-h_{20}\right) & i\left(h_{03}-h_{30}\right) & h_{03}+h_{30} \\ +i\left(h_{13}-h_{31}\right) & -i\left(h_{13}-h_{31}\right) & +i\left(h_{12}-h_{21}\right) & -h_{12}-h_{21} \end{pmatrix}$$

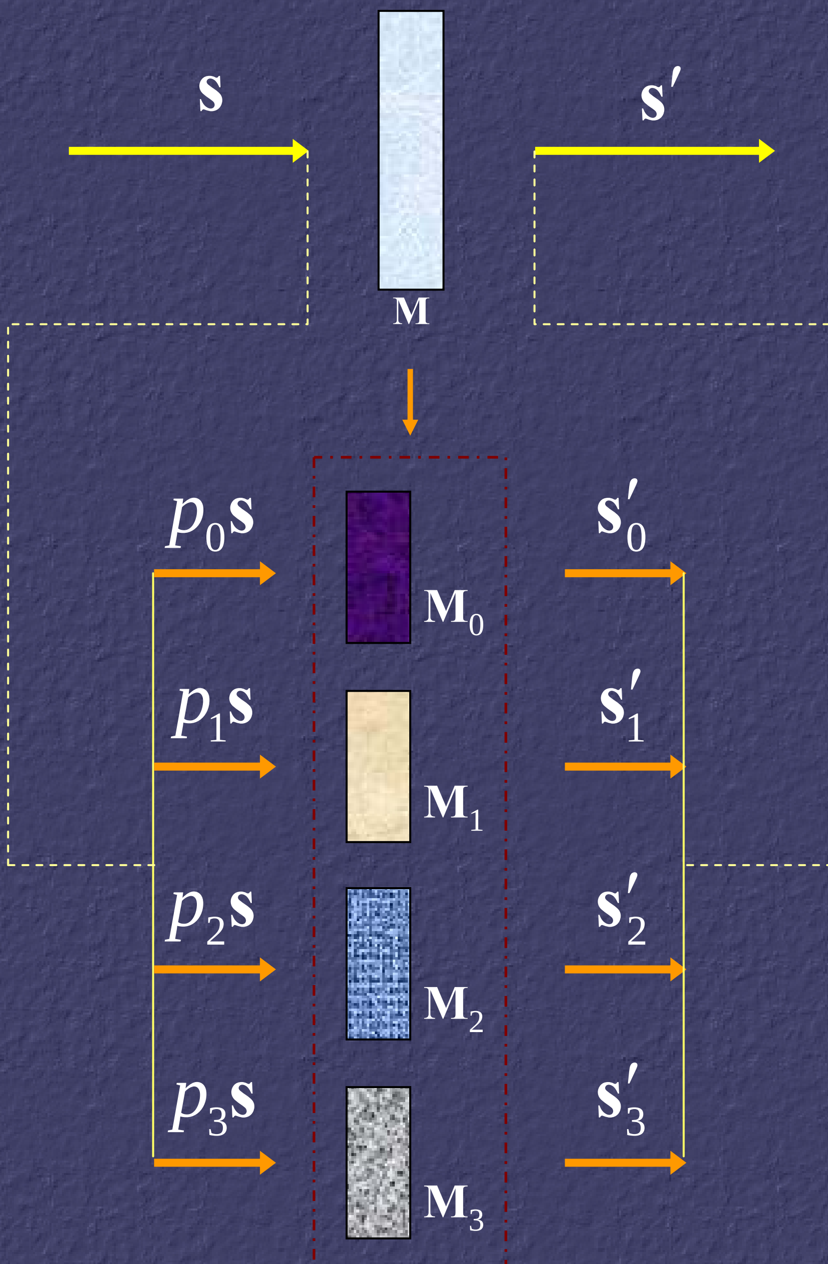
Serial and parallel decompositions

Serial decomposition



$$\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{M}_0$$

Parallel decomposition



2

Parallel decompositions

Spectral

Characteristic

Arbitrary

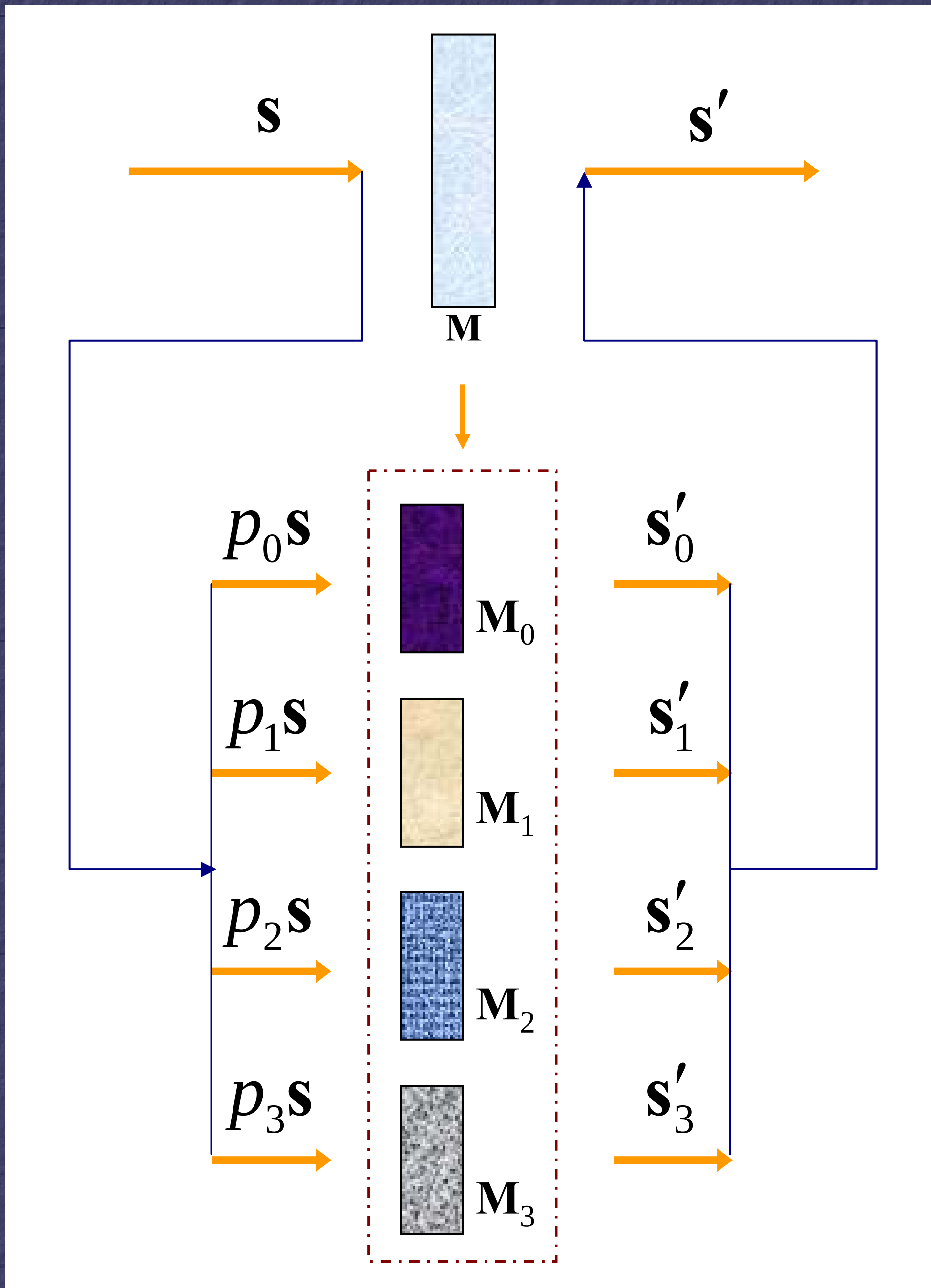
J. J. Gil, Eur. Phys. J. Appl. Phys. **40**, 1–47 (2007)

J. J. Gil, EPJ Web of Conferences **5**, 03001 (2010)

J. J. Gil, I. San José, R. Ossikovski, J. Opt. Soc. Am A **30**, 32-50 (2013)

J. J. Gil, I. San José, J. Opt. Soc. Am A **30**, 1078-1088 (2013)

Parallel decompositions



$$\mathbf{s}' = \mathbf{M}\mathbf{s}$$

$$\mathbf{s} = \sum_1 p_i \mathbf{s}$$

$$\mathbf{s}'_i = \mathbf{M}_i p_i \mathbf{s}$$

$$\mathbf{s}' = \sum \mathbf{s}'_i = \underbrace{\left(\sum p_i \mathbf{M}_i\right)}_{\mathbf{M}} \mathbf{s}$$

Spectral decomposition

Spectral decomposition

$$\mathbf{H} = \underbrace{\mathbf{U}\, diag\left(\lambda_0, \lambda_1, \lambda_2, \lambda_3\right)\mathbf{U}^{\dagger}}_{\begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}}$$

$$\mathbf{H} = \frac{\lambda_0}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}diag\left(\mathrm{tr}\mathbf{H},0,0,0\right)\mathbf{U}^{\dagger}\right]}^{\mathbf{H}_{J0}} + \frac{\lambda_1}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}diag\left(0,\mathrm{tr}\mathbf{H},0,0\right)\mathbf{U}^{\dagger}\right]}^{\mathbf{H}_{J1}} + \frac{\lambda_2}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}diag\left(0,0,\mathrm{tr}\mathbf{H},0\right)\mathbf{U}^{\dagger}\right]}^{\mathbf{H}_{J2}} + \frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}diag\left(0,0,0,\mathrm{tr}\mathbf{H}\right)\mathbf{U}^{\dagger}\right]}^{\mathbf{H}_{J3}}$$

Spectral decomposition

$$\mathbf{H} = \sum_{i=0}^{3} p_i \mathbf{H}_{Ji}$$

$$\mathbf{M}(\mathbf{H}) = \sum_{i=0}^{3} p_i \mathbf{M}_{Ji}(\mathbf{H}_{Ji})$$

$$\mathbf{H}_{Ji} \equiv (\mathrm{tr}\mathbf{H})(\mathbf{u}_i \otimes \mathbf{u}_i^{\dagger})$$

$$p_i \equiv \frac{\lambda_i}{\mathrm{tr}\mathbf{H}}$$

$$\sum_{i=0}^{3} p_i = 1$$

each term in the sum is generated by its corresponding eigenvector $\mathbf{u}_i$

Characteristic (or "trivial") decomposition

Characteristic decomposition

$$\mathbf{H} = \mathbf{U}\,\underbrace{diag\left(\lambda_0,\lambda_1,\lambda_2,\lambda_3\right)}\,\mathbf{U}^\dagger$$

$$(\lambda_0-\lambda_1)\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+(\lambda_1-\lambda_2)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}+(\lambda_2-\lambda_3)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}+\lambda_3\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$\Rightarrow$

$$\mathbf{H} = \frac{\lambda_0-\lambda_1}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}\,\mathrm{tr}\mathbf{H}\,diag\left(1,0,0,0\right)\mathbf{U}^\dagger\right]}^{\mathbf{H}_{J0}}+2\frac{\lambda_1-\lambda_2}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}\,\mathrm{tr}\mathbf{H}\,diag\left(1,1,0,0\right)\mathbf{U}^\dagger\right]}^{\mathbf{H}_1}+3\frac{\lambda_2-\lambda_3}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}\,\mathrm{tr}\mathbf{H}\,diag\left(1,1,1,0\right)\mathbf{U}^\dagger\right]}^{\mathbf{H}_2}+4\frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\overbrace{\left[\mathbf{U}\,\mathrm{tr}\mathbf{H}\,diag\left(1,1,1,1\right)\mathbf{U}^\dagger\right]}^{\mathbf{H}_3}$$

Characteristic decomposition of the Mueller matrix $\mathbf{M}$

$$P_\Delta\left(\mathbf{M}_i\right)=\sqrt{\frac{3-i}{3(1+i)}}$$

$$\mathbf{M}=\frac{\lambda_0-\lambda_1}{\mathrm{tr}\mathbf{H}}\mathbf{M}_0\left(\mathbf{H}_0\right)$$

$$+2\frac{\lambda_1-\lambda_2}{\mathrm{tr}\mathbf{H}}\mathbf{M}_1\left(\mathbf{H}_1\right)$$

$$+3\frac{\lambda_2-\lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{M}_2\left(\mathbf{H}_2\right)$$

$$+4\frac{\lambda_3}{\mathrm{tr}\mathbf{H}}\mathbf{M}_3\left(\mathbf{H}_3\right)$$

$$P_\Delta\left(\mathbf{M}_0\right)=1$$

$$P_\Delta\left(\mathbf{M}_1\right)=1/\sqrt{3}$$

$$P_\Delta\left(\mathbf{M}_2\right)=1/3$$

$$P_\Delta\left(\mathbf{M}_3\right)=0$$

Indices of purity and characteristic decomposition

Indices of polarimetric purity

$$P_1 \equiv \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{H}}\ ,\ P_2 \equiv \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{H}}\ ,\ P_3 \equiv \frac{\lambda_0 + \lambda_1 + \lambda_2 - 3\lambda_3}{\mathrm{tr}\mathbf{H}}$$

Degree of polarimetric purity

$$P_\Delta^2 = \frac{1}{3}\left(2P_1^2 + \frac{2}{3}P_2^2 + \frac{1}{3}P_3^2\right)$$

$$0 \le P_1 \le P_2 \le P_3 \le 1$$

- *Pure*: $P_\Delta = P_1 = P_2 = P_3 = 1$
- *Equiprobable mixture*: $P_\Delta = P_1 = P_2 = P_3 = 0$

Physical interpretation of the characteristic decomposition

in terms of the indices of polarimetric purity

$$\mathbf{M} = P_1 \mathbf{M}_{J0} + \left(P_2 - P_1\right)\mathbf{M}_1 + \left(P_3 - P_2\right)\mathbf{M}_2 + \left(1 - P_3\right)\mathbf{M}_3$$

$$\begin{cases} P_3 = 1 \Rightarrow \mathbf{M} = \mathbf{M}_{J0} + \left(P_2 - P_1\right)\mathbf{M}_1 + \left(1 - P_2\right)\mathbf{M}_2 \\ P_2 = 1 \Rightarrow \mathbf{M} = \mathbf{M}_{J0} + \left(P_2 - P_1\right)\mathbf{M}_1 \\ P_1 = 1 \Rightarrow \mathbf{M} = \mathbf{M}_{J0} \end{cases}$$

Arbitrary decomposition

Arbitrary decomposition: existence

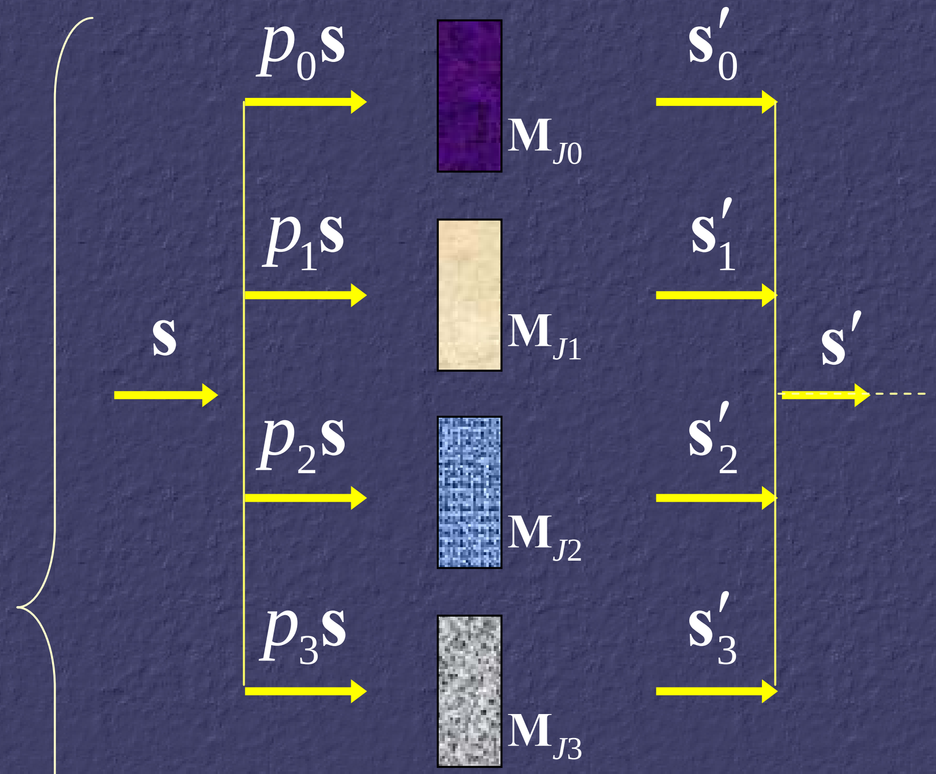
There exist decompositions of $\mathbf{M}$ into pure components, other than the spectral decomposition?

Arbitrary decomposition: existence

1. Take a set of arbitrary pure elements $\mathbf{M}_{J1}$, $\mathbf{M}_{J2} \dots \mathbf{M}_{Jn}$

2. Mix them into a parallel combination

3. $$\mathbf{M} \equiv \sum_{i=1}^{n} p_i \mathbf{M}_{Ji}$$



$$\sum_{i=1}^{n} p_i = 1$$

4. **Obviously, $\mathbf{M}_{Ji}$ are arbitrary and not necessarily coincide with the "spectral" components**

Arbitrary decomposition

How many "arbitrary decompositions"
do exist?

$$\infty$$

Arbitrary decomposition

$$\mathbf{H} = \sum_{i=0}^{k-1} p_i \mathbf{H}_{Ji}, \quad k \equiv \operatorname{rank}(\mathbf{H})$$

$\operatorname{tr}\mathbf{H}_{Ji} = m_{00} = \operatorname{tr}\mathbf{H}$ same mean transmittance

$\operatorname{rank}(\mathbf{H}_{Ji}) = 1:$ nondepolarizing components

$\sum_{i=0}^{k-1} p_i = 1$ incoherent convex sum

$k:$ minimum number of pure $\parallel$ components

Arbitrary decomposition procedure

1. Mueller Matrix $\mathbf{M}(\mathbf{H})$, $k = \text{rank}\,(\mathbf{H})$

2. Choose a set of k independent unit vectors $\mathbf{w}_i \in \text{range}\,(\mathbf{H})$

3. $\mathbf{H}_{Ji} = m_{00}\left(\mathbf{w}_i \otimes \mathbf{w}_i^{\dagger}\right)$

4. $p_i = \dfrac{1}{m_{00}}\,\text{tr}\left\{\begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix}\left[\mathbf{w}_i \otimes \mathbf{w}_i^{\dagger}\right]\right\}$

$$\mathbf{H} = \sum_{i=0}^{k-1} p_i \mathbf{H}_{Ji}$$

$$\mathbf{M} = \sum_{i=0}^{3} p_i \mathbf{M}_{Ji}\left(\mathbf{H}_{Ji}\right)$$

The spectral decomposition is only a particular case of the arbitrary decomposition

3

Polarimetric subtraction

J. J. Gil, I. San Jose, “Polarimetric subtraction of Mueller matrices,” J. Opt. Soc. Am. A, **30**, 1078-1088 (2013)

Statement of the problem

Given the Mueller matrices of:

- the sample as a whole $\mathbf{M}_k\,(\mathbf{H}_k)$, $\text{rank}\,(\mathbf{H}_k) = k$
- a known component $\mathbf{M}_m\,(\mathbf{H}_m)$, $\text{rank}\,(\mathbf{H}_m) = m < k$

Find p such that

$$\mathbf{M}_k = p\mathbf{M}_m + (1-p)\,\mathbf{M}_X\,, \quad (0 < p < 1)$$

$$\mathbf{M}_X = (\mathbf{M}_k - p\mathbf{M}_m)/(1-p)$$

↓

"difference Mueller matrix"

Condition of subtractability

$$\mathrm{range}\left(\mathbf{H}_m\right) \subseteq \mathrm{range}\left(\mathbf{H}_k\right)$$

Subspace generated by the eigenvectors of $\mathbf{H}_m$ with non-zero eigenvalues

Subspace generated by the eigenvectors of $\mathbf{H}_k$ with non-zero eigenvalues

Subtraction procedure: pure subtrahend

$\mathbf{M}_k\,(\mathbf{H}_k),\ \mathbf{M}_0\,(\mathbf{H}_0)$

$\mathrm{range}\left(\mathbf{H}_0\right) \subseteq \mathrm{range}\left(\mathbf{H}_k\right)$?

No

Unphysical hypothesis

$\mathbf{M}_0 \not\subset \mathbf{M}_k$

END

Yes

Calculate p

$$\mathbf{M}_X = \frac{\mathbf{M}_k - p\mathbf{M}_0}{1-p}$$

$$p = \frac{1}{m_{00}^2}\,\mathrm{tr}\left\{ \begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \mathbf{H}_0 \right\}$$

Subtraction procedure: depolarizing subtrahend

$\mathbf{M}_k\,(\mathbf{H}_k),\ \mathbf{M}_m\,(\mathbf{H}_m)$

$\mathrm{range}\left(\mathbf{H}_m\right) \subseteq \mathrm{range}\left(\mathbf{H}_k\right)$?

No

Unphysical hypothesis

$\mathbf{M}_m \not\subset \mathbf{M}_k$

END

Yes

Calculate p

$$\mathbf{M}_X = \frac{\mathbf{M}_k - p\mathbf{M}_m}{1-p}$$

J. J. Gil, I. San Jose, "Polarimetric subtraction of Mueller matrices," J. Opt. Soc. Am. A, **30**, 1078-1088 (2013)

The arbitrary decomposition and the polarimetric subtraction can also be applied to Stokes vectors

[J. J. Gil, Eur. Phys. J. Appl. Phys. **40**, 1–47 (2007)]

$$\mathbf{s} = I\begin{bmatrix} 1 \\ P\mathbf{u} \end{bmatrix} = p\mathbf{s}_1 + (1-p)\mathbf{s}_2$$

$$\left\{ \quad \mathbf{s}_1 \equiv I\begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix}, \quad \mathbf{s}_2 \equiv I\begin{bmatrix} 1 \\ \mathbf{w} \end{bmatrix}; \quad |\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1 \quad \right\}$$

$$p = \frac{1-P^2}{2\left(1-P\mathbf{u}^T\mathbf{v}\right)}; \quad \mathbf{w} = \frac{P\mathbf{u}-p\mathbf{v}}{1-p}$$

4

Physical quantities in a Mueller matrix

Arrow decomposition of $\mathbf{M}$

$$\mathbf{M}_A(\mathbf{M}) \equiv \mathbf{M}_{RO}^T \mathbf{M} \mathbf{M}_{RI}^T = m_{00} \begin{pmatrix} 1 & \mathbf{D}_A^T \\ \mathbf{P}_A & \mathrm{diag}(l_1, l_2, l_3) \end{pmatrix}$$

$$l_1 \geq l_2 \geq l_3 : \textit{singular values of } \mathbf{m}(\mathbf{M})$$

$$\mathbf{D}_A = \mathbf{m}_{RI} \mathbf{D}$$

$$\mathbf{P}_A = \mathbf{m}_{RO}^T \mathbf{P}$$

16 quantities from $\mathbf{M}$

$$m_{00}(1), \mathbf{M}_{RI}(3), \mathbf{M}_{RO}(3), \mathbf{P}(3), \mathbf{D}(3), l_i, l_2, l_3(3)$$

A set of 6 meaningful independent invariant quantities of $\mathbf{M}$

Mean transmittance *(transmittance for unpolarized light)*	$0 \leq m_{00} \leq 1$
Polarizance	$0 \leq P \leq 1$
Diattenuation	$0 \leq D \leq 1$
Indices of purity	$0 \leq P_1 \leq P_2 \leq P_3 \leq 1$

Complete set of invariant quantities

Independent quantities	**Dependent quantities**
m_{00}	
P	P_Δ *Depolarization index*
D	
P_1	P_S *Degree of spherical purity*
P_2	$P_P \equiv \sqrt{\left(D^2 + P^2\right)/2}$ *Degree of polarizance*
P_3	

Dependent invariant quantities

$$P_{\Delta}^{2} = \frac{1}{3}\left(2P_{1}^{2} + \frac{2}{3}P_{2}^{2} + \frac{1}{3}P_{2}^{2} \right)$$

$$P_{P}^{2} = \frac{1}{2}\left(D^{2} + P^{2}\right)$$

$$P_{S}^{2} = P_{\Delta}^{2} - \frac{1}{3}\left(D^{2} + P^{2}\right) = P_{\Delta}^{2} - \frac{2}{3}P_{P}^{2}$$

Indices of polarimetric purity

Degree of polarization	$P_1 \equiv \dfrac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{H}}$
Degree of directionality	$P_2 \equiv \dfrac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{H}}$
Degree of 4D-directionality	$P_3 \equiv \dfrac{\lambda_0 + \lambda_1 + \lambda_2 - 3\lambda_3}{\mathrm{tr}\mathbf{H}}$

$$P_\Delta = \sqrt{\frac{2P_1^2 + \frac{2}{3}P_2^2 + \frac{1}{3}P_3^2}{3}}$$

$$0 \le P_1 \le P_2 \le P_3 \le 1$$

Pure:

$$P_\Delta = P_1 = P_2 = P_3 = 1$$

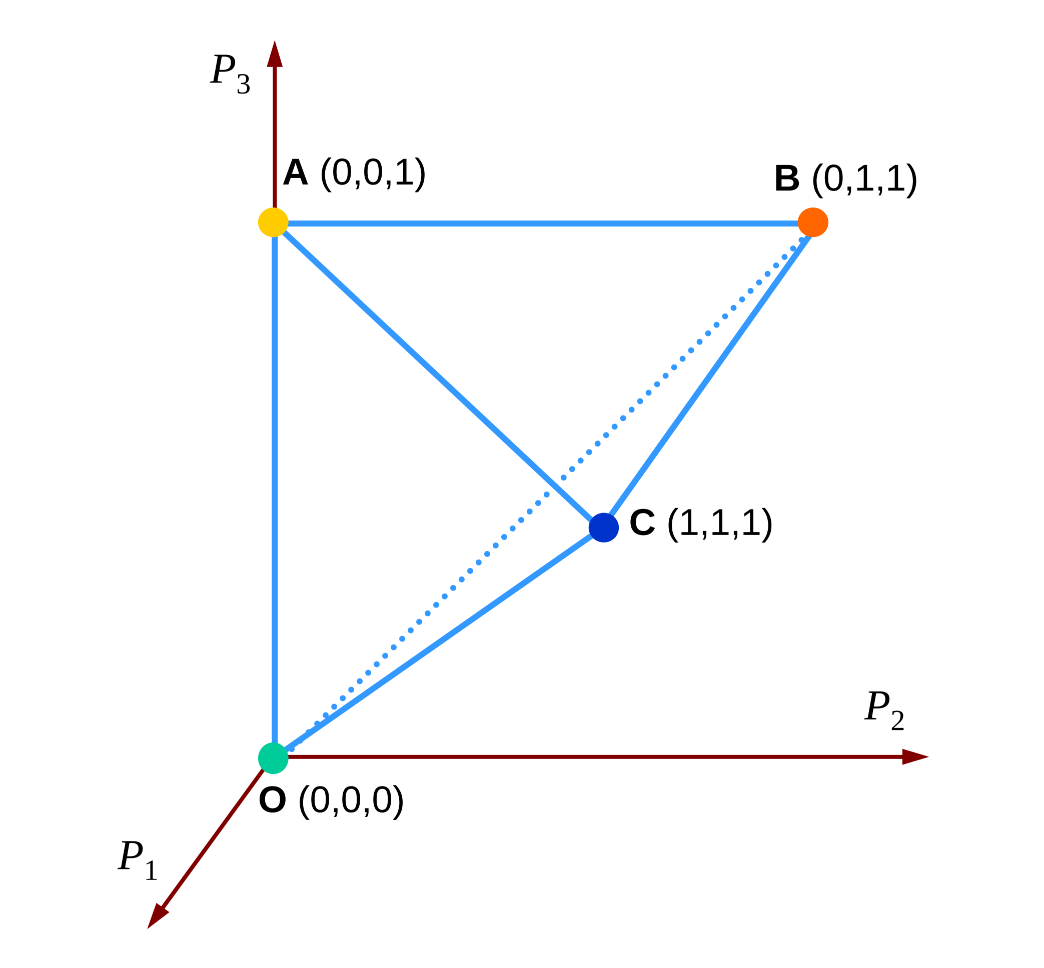
Equiprobable mixture:

$$P_\Delta = P_1 = P_2 = P_3 = 0$$

Indices of polarimetric purity for polarized light and for media

	Light	Light	Medium
Dim.	**2D**	**3D**	**4D**
Coherency matrix	$\mathbf{\Phi} = \frac{1}{2}\sum_{i=0}^{3} s_i \boldsymbol{\sigma}_i$	$\mathbf{R} = \frac{1}{3}\sum_{i=0}^{8} q_i \mathbf{\Omega}_i$	$\mathbf{H} = \frac{1}{4}\sum_{i,j=0}^{3} m_{ij} \mathbf{E}_{ij}$
Purity quantities	$P = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{\Phi}}$	$P_1 = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{R}}$ $P_2 = \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{R}}$	$P_1 = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{R}}$ $P_2 = \frac{\lambda_0 + \lambda_1 - 2\lambda_2}{\mathrm{tr}\mathbf{R}}$ $P_2 = \frac{\lambda_0 + \lambda_1 + \lambda_2 - 3\lambda_3}{\mathrm{tr}\mathbf{R}}$
Limits	$0 \le P \le 1$	$0 \le P_1 \le P_2 \le 1$	$0 \le P_1 \le P_2 \le P_3 \le 1$
Global purity	$P_{(2)} \equiv P = \frac{\lambda_0 - \lambda_1}{\mathrm{tr}\mathbf{\Phi}}$	$P_{(3)} = \frac{1}{2}\sqrt{3P_1^2 + P_2^2}$	$P_{(4)} = \frac{1}{\sqrt{3}}\sqrt{2P_1^2 + \frac{2}{3}P_2^2 + \frac{1}{3}P_3^2}$

Feasible region for the indices of purity



$$0 \le P_1 \le P_2 \le P_3 \le 1$$

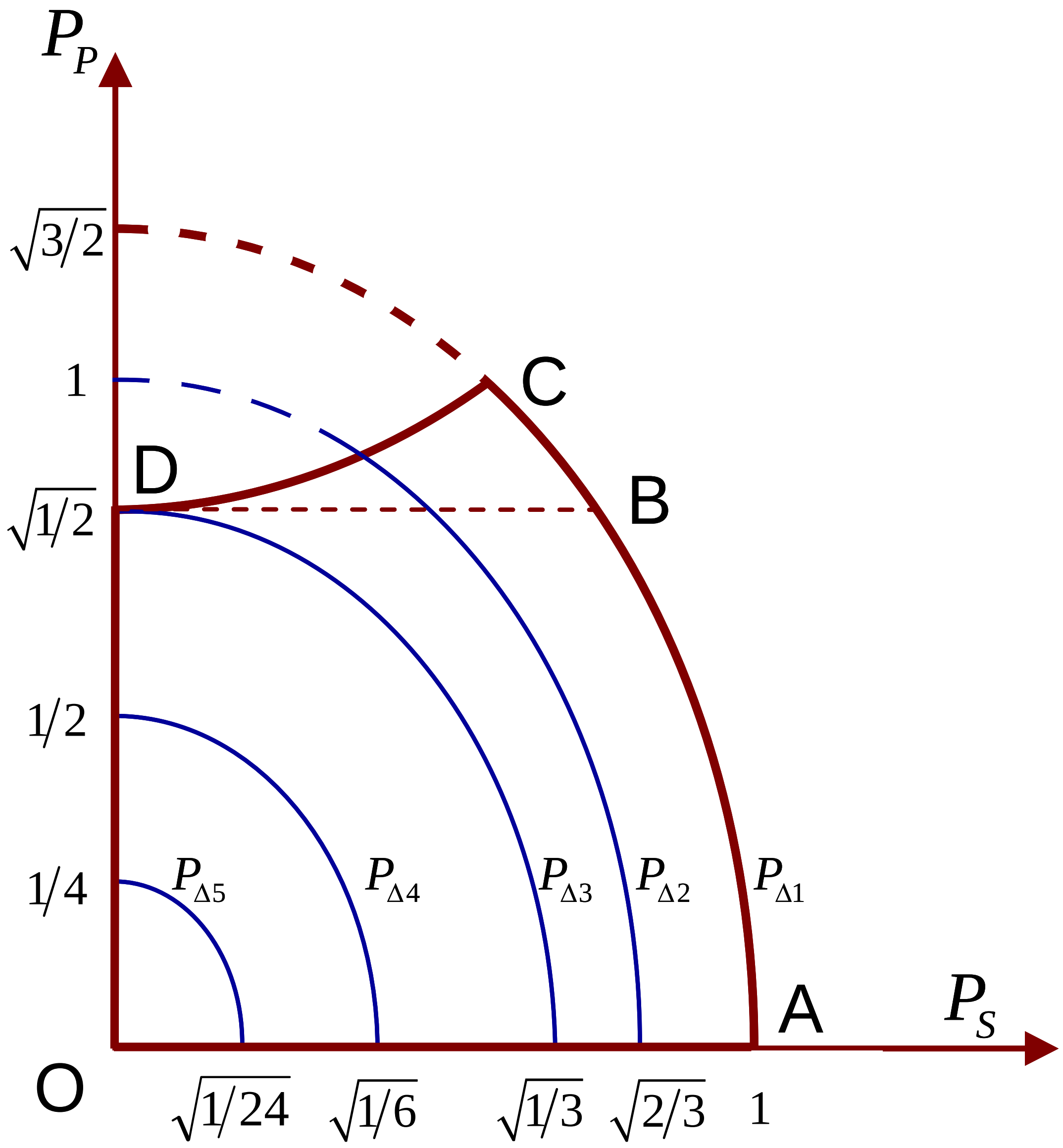
C *Pure:*

$$P_\Delta = P_1 = P_2 = P_3 = 1$$

O *Equiprobable mixture:*

$$P_\Delta = P_1 = P_2 = P_3 = 0$$

Feasible region for P_P and P_S



Thank you!

***More information*:**

J. J. Gil, "Review on Mueller matrix algebra for the analysis of polarimetric measurements," Journal of Applied Remote Sensing **8**(1), 081599 (2014)

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